Experimental determination of the Boltzmann constant by Johnson noise

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Johnson noise, an electronic noise created by the thermal agitation of charge carriers in a conductor, is used to empirically determine the Boltzmann constant. Using a shielded set of resistors and a digital FFT spectrum analyzer, we report a value of $k_B = (1.36 \pm 0.05) * 10^{-23}\ [\frac{J}{K}]$ - within one standard deviation of published values.

Introduction

Johnson-Nyquist noise is an inherent random fluctuation in voltage arising from the irregular thermal motion of charge carriers in an electrical conductor at equilibrium. In 1928[1], Johnson experimentally demonstrated that the root mean square voltage across a conductor is proportional to the resistance and absolute temperature and does not depend on the physical or chemical makeup of the conductor. We examine the behavior of Johnson noise as a function of electrical resistance by analyzing a region of white noise using a digital FFT spectrum analyzer. Based on the measured root mean square voltage (V_{rms}) and resistor temperature, we empirically determine Boltzmann's constant, k_B.

The fundamental relation between V_{rms}, temperature (T), resistance (R) and frequency range (B) can be determined by the equipartition theorem of statistical mechanics that relates the temperature of a system with its average energies[2]. Thus, we have

$$V_{rms} = \sqrt{4Rk_BTB} \qquad [1]$$

In our experiment, the variable is resistance while the bandwidth and temperature remain fixed. By measuring power spectral density (PSD) in units of $V/\sqrt{Hz}$ for a set bandwidth, we determine the V_{rms} for a particular resistance and can use a range of resistors to determine k_B.

PSD is proportional to frequency by the relationship $1/f^{\alpha}$, where for white noise α approaches zero, but can range from $\in(0,2)$. Measurements of PSD relevant to Johnson noise must be made in the white noise region. At low frequencies, the PSD follows a $1/f^{1}$ relationship. However there exists a point at which $1/f$ noise becomes less than white noise and PSD is dominated by white noise (Figure 1). It is this region that relates to the fundamental noise generated by thermal motion of charge carriers – Johnson noise.

Note that in the case of an applied voltage across the conductor, PSD distribution changes. This is related to the emergence of shot-noise, an artifact of the quantized nature of particles. This experiment minimizes contributions from shot-noise by placing the resistance elements in shielded box.

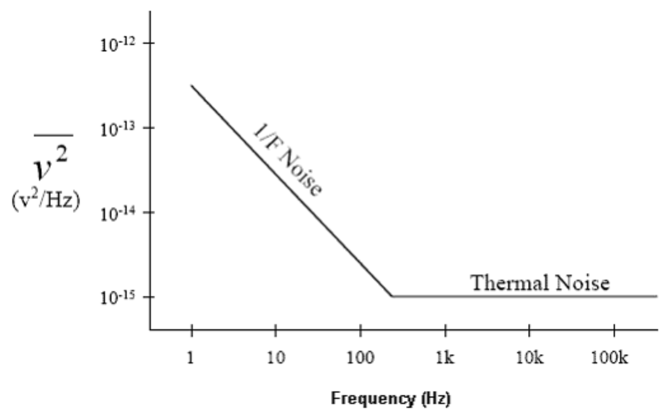

Figure 1 - Power spectral density as a function of frequency[3]

Experimental Methods

The laboratory setup is depicted in Figure 2. The 'Johnson noise box' consists of a shielded metallic box, sealed with copper tape that encloses a range of resistors (10, 20, 51.1, 100, 200, 499 and 1000 $k\Omega$). Shielding insures that the noise, as measured by the digital FFT analyzer, represents only Johnson noise and not shot noise created by an induced current from a stray signal.

The resistive elements are connected to the input of a digital FFT spectrum analyzer that outputs PSD as a function of frequency. PSD is measured with a bandwidth of 31.25 Hz at a frequency of 5.5 kHz, well into the white noise region of Figure 1. Each PSD data point in Table 1 represents the average of two 400 cycle measurements.

Resistor temperature was monitored using a type E thermocouple that we calibrated with 200 ml of boiling water and ice water to an accuracy of ± 1 Kelvin. For the duration of the experiment, the temperature of each resistor remained constant at 298± 1 K.

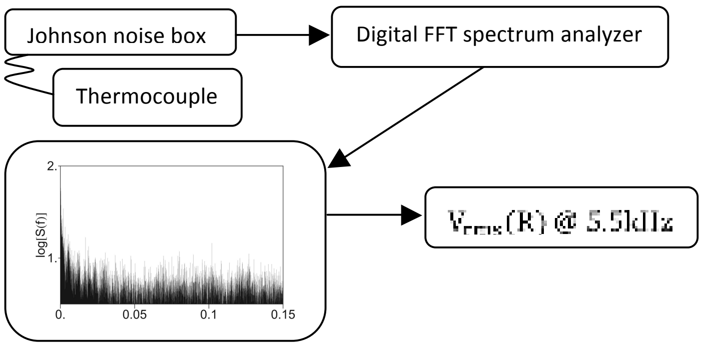

Figure 2 – Experimental setup

Results and Data Analysis

Initial data showed a Johnson noise significantly below the theoretical predictions, but further analysis revealed a fundamental systematic error in the experiment – the spectrum analyzer's input impedance. The external resistor from the Johnson noise box is in fact parallel with the input resistance yielding an effective resistance less than originally thought:

$$R_{eff} = \frac{R_{box}R_{input}}{R_{box} + R_{input}}$$

It follows that with an input capacitance, C, of 15pF and resistance, R, of 1 MΩ, the impedance of the spectrum analyzer can be expressed as a function of the angular frequency ($\omega = 2\pi f$) of the circuit using the relationship $V^2 = I^2Z^2$.

$$Z = \frac{1}{\frac{1}{R} + i\omega C}$$

It can be shown[4] that this results in an effective amplification that can occur in the system as a function of frequency. This gain term, $g(f)$, can then replace B in [1] by integrating over all frequencies. The resulting substitution is:

$$B = \int_0^\infty \frac{g(f)^2}{1 + (2\pi f CR)^2} df$$

This term could not be quantitatively evaluated in this experiment, but is likely a primary source of systematic error.

Table 1 reflects how PSD was observed to behave as a function of resistance. From the PSD measurements, we compute V_{rms}, and hence k_B, by the relationship:

$$V_{rms} = \sqrt{PSD} * \sqrt{B} = \sqrt{4Rk_BTB}$$

$R\ [k\Omega]$	$R_{eff}\ [k\Omega]$	$PSD\ \left[\frac{nV_{rms}}{\sqrt{Hz}}\right]$	nV_{rms}
10.0±0.1	9.9±0.1	16.4±1	22.6±1
51.1±0.1	48.6±0.1	22.6±1	26.5±1
100.0±0.1	90.9±0.1	32.8±1	31.9±1
200.0±0.1	166.7±0.1	49.7±1	39.4±1
499.0±0.1	332.9±0.1	67.6±1	45.9±1
1000.0±0.1	500.0±0.1	84.9±1	51.5±1

Table 1 - PSD and Vrms at 5.5kHz as a function of effective resistance

By propagation of errors for resistor values and PSD measurements, these data result in $k_B = (1.36 \pm 0.05) * 10^{-23} \left[\frac{J}{K}\right]$. Comparing this to the published[5] value of $k_B = 1.3806504(24) * 10^{-23} \left[\frac{J}{K}\right]$ it would appear that Johnson noise provides a relatively accurate method of determining Boltzmann's constant.

It should be noted however, that we encountered difficulty in getting consistent PSD data from the spectrum analyzer on consecutive days and ultimately used only one of our two data sets for analysis. The type E thermocouple used also had significant errors ($\pm 1K$) and was not the ideal instrument for the measurement of resistor temperatures, but was the best available. Additionally, errors in the input impedance of the spectrum analyzer were not accounted for, but likely had little contribution to the overall error.

Conclusions

Our measured value of Boltzmann's constant, $k_B = (1.36 \pm 0.05) * 10^{-23} \left[\frac{J}{K}\right]$, is in agreement with published values and confirms that Johnson noise has the potential to return a fairly precise value for this constant. Future directions for this experiment could include a quantitative assessment of the gain factor introduced in [1] and more accurate measurements of the resistor temperatures. An alternative experiment could exploit variations in temperature to empirically determine Boltzmann's constant.

References

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